Problem set 1 for 41076: Methods in Quantum Computing

due August 26th, at 2 pm
15 regular + 2 bonus points

1 Linear algebra refresher (7 points)

Let us consider U to be a 2-by-2 unitary matrices such that $\det U = 1$.

1. Let us write $U = (\vec{u}\ \vec{v})$ where $\vec{u}, \vec{v}$ correspond to the columns of matrix U. Show that if U is unitary, $\vec{u}, \vec{v}$ must be orthonormal (both vectors have norm 1 and are orthogonal to each other).

   Note: One could use the exact same steps to show that the rows are also orthonormal (this is not required in this problem set). This means that if we write $U = \begin{pmatrix} \vec{a}^T \\ \vec{b}^T \end{pmatrix}$ where $\vec{a}^T, \vec{b}^T$ are row vectors, $\vec{a}^T, \vec{b}^T$ are also orthonormal.

2. Show that any 2-by-2 unitary matrix with determinant 1 can be expressed as

$$\begin{pmatrix} e^{i(\alpha/2+\beta/2)} \cos\frac{\theta}{2} & e^{i(\alpha/2-\beta/2)} \sin\frac{\theta}{2} \\ -e^{i(-\alpha/2+\beta/2)} \sin\frac{\theta}{2} & e^{i(-\alpha/2-\beta/2)} \cos\frac{\theta}{2} \end{pmatrix} \tag{1}$$

   where α, β, θ are real-valued. This means not only that matrix (1) is unitary but also that this is the most general form of a unitary matrix. Part 1 might be helpful for this.

3. Define single qubit rotation gates

$$R_z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \quad R_y(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}. \tag{2}$$

   Show that the matrix in Eq. (1) can be implemented using three rotations as $R_z(\alpha)R_y(\theta)R_z(\beta)$.

2 Quantum circuits (4 points)

1. Define $\mathrm{Ph}(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$ to be a global phase gate (also known as phase shift). Show that there is always a gate $E(\delta)$ such that the identity in Fig. 1 holds.

2. Suppose we implemented the circuit in Fig. 2 and measured the first register. If we measure the first qubit to be in a state $|0\rangle$, what is the state of the second register? The white-control notation means that the control is activated if the qubit is in state 0 while the black control is a standard controlled gate, activated if the control qubit is in state 1.

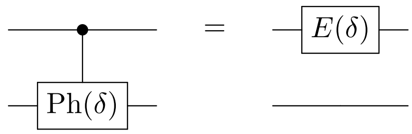

Figure 1: Prove that controlled global phase can be written as a single qubit gate on the control register.

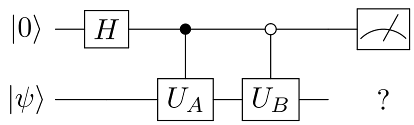

Figure 2: What is the state of the second register if the state of the first register was measured to be $|0\rangle$?

3 Working with pure and mixed states (6 points)

Alice and Bob share the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$.

1. Verify that $|\langle\psi||\psi\rangle| = 1$
2. Compute the density matrix $\sigma = |\psi\rangle\langle\psi|$.
3. Compute the purity $\text{Tr}(\sigma^2)$. Is σ pure?
4. Compute the density matrix of Alice's state $\sigma_A = \text{Tr}_B(\sigma)$.
5. Compute the purity of Alice's state. Is Alice's state pure?
6. Is $|\psi\rangle$ an entangled state and why/why not?