41076: Methods in Quantum Computing

Quantum channels, measurements and tomography

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Abstract

Contents to be covered in this lecture are

1. The no-cloning theorem
2. Distance measures
3. Quantum channels
4. Noise channels
5. Measurement

In the last lecture we covered quantum states and unitary operations. Today we will continue with a description of a more general form of operations.

1 The no-cloning theorem

A non-intuitive property of quantum mechanics that would be able to copy a general (unknown) quantum state. This is in stark contrast with classical information that can be always copied.

Suppose that we have two quantum systems of equal size $\mathcal{H}_A = \mathcal{H}_B$. Given a quantum state $|\phi\rangle_A \in \mathcal{H}_A$, if quantum mechanics allows the operation of 'copying', then this copying operation U_{copy} should achieve

$$U_{\text{copy}}(|\phi\rangle_A \otimes |0\rangle_B) = |\phi\rangle_A \otimes |\phi\rangle_B. \tag{1}$$

In other words, the copying operation should produce a second copy of $|\phi\rangle$ in $\mathcal{H}_B$ (that was initially prepared in some ground state $|0\rangle_B$.)

Theorem 1 (No-Cloning theorem)**.** *There is no unitary operation U_{copy} on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that for all $|\psi\rangle_A \in \mathcal{H}_A$ and $|0\rangle_B \in \mathcal{H}_B$*

$$U_{\text{copy}}(|\phi\rangle_A \otimes |0\rangle_B) = e^{if(\phi)}|\phi\rangle_A \otimes |\phi\rangle_B \tag{2}$$

for some number $f(\phi)$ that depends on the initial state $|\phi\rangle$.

Exercise 2. *Prove the no-cloning theorem by contradiction.*

a Assuming U_{copy} exists, take two states $|\phi_A\rangle$ and $|\psi\rangle$. Now apply U_{copy} on both of them and compute the resulting inner product

$(\langle\phi|_A \otimes \langle 0|_B)U_{\text{copy}}^\dagger U_{\text{copy}}(|\psi\rangle_A \otimes |0\rangle_B)$.

b Explain how (a) leads to a contradiction.

1.1 Proof of the non-cloning theorem

Assume such a coping operation exists. Then for any two states $|\psi\rangle_A, |\phi\rangle_A \in \mathcal{H}_A$, the following holds

$$\begin{aligned} U_{\text{copy}}(|\phi\rangle_A \otimes |0\rangle_B) &= e^{if(\phi)}|\phi\rangle_A \otimes |\phi\rangle_B & (3) \\ U_{\text{copy}}(|\psi\rangle_A \otimes |0\rangle_B) &= e^{if(\psi)}|\psi\rangle_A \otimes |\psi\rangle_B. & (4) \end{aligned}$$

Now

$$\begin{aligned} (\langle\phi|_A \otimes \langle 0|_B)U^\dagger_{\text{copy}}U_{\text{copy}}(|\psi\rangle_A \otimes |0\rangle_B) &= \langle\phi|\psi\rangle_A & (5) \\ &= e^{i(f(\psi)-f(\phi))}\langle\phi|\psi\rangle_A\langle\phi|\psi\rangle_B. & (6) \end{aligned}$$

The first equality follows because $U^\dagger_{\text{copy}}U_{\text{copy}} = I$ and $\langle 0|0\rangle_B = 1$. Hence

$$|\langle\phi|\psi\rangle_A|^2 = |\langle\phi|\psi\rangle_A|, \tag{7}$$

which implies that either $|\langle\phi|\psi\rangle_A| = 1$ or $|\langle\phi|\psi\rangle_A| = 0$. This allows us to conclude that not a single universal copying operation U_{copy} exists for two arbitrary states.

2 Distance Measures

2.1 Matrix Norm

We will introduce a few useful matrix norms in this section. First of all, every norm $\|\cdot\|$ must satisfy the following conditions.

- $\|A\| \geq 0$ with equality if and only if $A = 0$.
- $\|\alpha A\| = |\alpha|\|A\|$ for any $\alpha \in \mathbb{C}$.
- Triangle inequality: $\|A + B\| \leq \|A\| + \|B\|$.

Definition 3 (Schatten norm)**.** *For $p \in [1, \infty)$, the Shatten p-norm of a matrix $A \in \mathbb{C}^{m\times n}$ is defined as*

$$\|A\|_p := \text{Tr}(|A|^p)^{\frac{1}{p}} \tag{8}$$

where $|A| := \sqrt{A^\dagger A}$. We extend $p \to \infty$ as follows

$$\|A\|_\infty := \max\{\|A\boldsymbol{x}\| : \forall \boldsymbol{x} \in \mathbb{C}^n,\ \|\boldsymbol{x}\| = 1\}. \tag{9}$$

Properties of Schatten p-norms are summarized below

1. The Schatten norms are unitarily invariant: for any unitary operators U and V

   $$\|UAV\|_p = \|A\|_p \tag{10}$$

   for any $p \in [1, \infty]$.

2. The Schatten norms satisfy Hölder's inequality: for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times \ell}$, it holds that

$$\|AB\|_1 \leq \|A\|_p \|B\|_q, \tag{11}$$

where $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3. Sub-multiplicativity: for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times \ell}$, it holds that

$$\|AB\|_p \leq \|A\|_p \|B\|_p. \tag{12}$$

4. Monotonicity: for $1 \leq p \leq q \leq \infty$, it holds that

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|_\infty. \tag{13}$$

Exercise 4. *Denote by $\sigma_i(A)$ the i-th (non-zero) singular value of A. Show that*

$$\|A\|_p = \left(\sum_i (\sigma_i(A))^p\right)^{\frac{1}{p}}. \tag{14}$$

There are important special cases of Schatten p-norm. Specifically, the Schatten 1-norm is commonly known as the *trace norm*, and will lead to the definition of trace distance in Sec. 2.2. The Schatten 2-norm is also known as the *Frobenius norm* whose explicit form is given below.

Definition 5 (Frobenuis norm)**.** *The Frobenius norm (or the Hilbert-Schmidt norm) of a matrix $A \in \mathbb{C}^{m\times n}$ is defined as*

$$\|A\|_2 \equiv \|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |A_{i,j}|^2}. \tag{15}$$

Finally, the Schatten ∞-norm is also called the *operator norm* or the *spectral norm* whose definition is given in Eq. (9).

2.2 Trace Distance and Fidelity

We will introduce two commonly used distance measures in quantum information science; namely the trace distance and fidelity.

Definition 6 (Trace Distance)**.** *The trace distance between two operators A and B is given by*

$$\|A - B\|_1 := \mathrm{Tr}\,|A - B|.$$

Exercise 7.

$$\|\sigma - \rho\|_1 = \max_{-I \leq \Lambda \leq I} \mathrm{Tr}[\Lambda(\sigma - \rho)]. \tag{16}$$

Denote $T(\rho, \sigma) \equiv \|\rho - \sigma\|_1$. The trace distance of two density operators is an extension of total variation distance of probability measures:

$$T(P, Q) = \frac{1}{2}\sum_x |p(x) - q(x)|, \tag{17}$$

where P and Q are probability distributions with pdf $p(x)$ and $q(x)$, respectively.

Properties of the trace distance include

- $T(\rho, \sigma) = 0$ if and only if $\rho = \sigma$.
- Invariant under unitary operation: $T(U\rho U^\dagger, U\sigma U^\dagger) = T(\rho, \sigma)$
- Contraction: $T(\mathcal{N}(\rho), \mathcal{N}(\sigma)) \leq T(\rho, \sigma)$, where $\mathcal{N}$ is any trace-preserving and completely positive map.
- Convexity: $T(\sum_i p_i \rho_i, \sigma) \leq \sum_i p_i T(\rho_i, \sigma)$.

Definition 8 (Fidelity)**.** *For $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, their fidelity is*

$$F(\rho, \sigma) := \left(\operatorname{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2.$$

Note that fidelity is not a metric on $\mathcal{D}(\mathcal{H})$.

Exercise 9. *Shot that, for $\rho, \sigma \in \mathcal{D}(\mathcal{H})$,*

$$F(\rho, \sigma) = \min_{\Lambda_i} \left(\sum_i \sqrt{\operatorname{Tr}[\rho\Lambda_i]\operatorname{Tr}[\sigma\Lambda_i]} \right)^2 \tag{18}$$

where $\Lambda = \{\Lambda_i\}$ is a POVM [2].

Properties of the fidelity include

- Symmetry: $F(\rho, \sigma) = T(\sigma, \rho)$.
- $0 \leq F(\rho, \sigma) \leq 1$.
- $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma)$.
- $F(|\psi_\rho\rangle, |\psi_\sigma\rangle) = |\langle\psi_\rho|\psi_\sigma\rangle|^2$.
- $F(\mathcal{N}(\rho), \mathcal{N}(\sigma)) \geq F(\rho, \sigma)$, where $\mathcal{N}$ is any trace-preserving and completely positive map.

3 Quantum Channels

The most general operation on quantum states is a quantum channel, also known as completely positive trace-preserving map (CPTP map). The ensures that quantum states will always get mapped onto valid quantum states. This condition is even more complicated than it sounds; if we apply a quantum channel on only a part of a quantum state, we still must get a valid density matrix after the transformation. Any such channel can be written as

$$\Phi(\sigma) = \sum_i B_i \sigma B_i^\dagger \quad \text{where} \quad \sum_i B_i^\dagger B_i = \mathbf{1}. \tag{19}$$

This is known as the Kraus representation and the operators B_i as Kraus operators.

Exercise 10. *Show that transpose is a positive map, but not a completely positive map.*

Definition 11. *A quantum channel $\mathcal{N}$ is unital if $\mathcal{N}(I) = I$.*

Examples

- Dephasing Channel:
$$\mathcal{N}(\rho) = (1-p)\rho + pZ\rho Z.$$
The Kraus operators are $B_1 = \sqrt{1-p}I$ and $B_2 = \sqrt{p}Z$.

- Depolarizing Channel:
$$\mathcal{N}(\rho) = (1-p)\rho + p\pi,$$
where π is the completely mixed state.

- Pauli Channel:
$$\mathcal{N}(\sigma) = \sum_{i,j=0}^{1} p(i,j) Z^i X^j \sigma X^j Z^i$$
where we denote $X^0 = Z^0 = I$.

- Measure-and-prepare channel: For a POVM $\{\Lambda_i\}$ and a collection of quantum states $\{\sigma_i\}$, we can define
$$\mathcal{N}(\rho) = \sum_i \sigma_i \operatorname{Tr}(\Lambda_i \rho). \tag{20}$$
This channel is also known as an *entanglement-breaking* channel.

Exercise 12. *The set of generalized Pauli matrices $\{U_m\}_{m\in[d^2]}$ is defined by $U_{l\cdot d+k} = \hat{Z}_d(l)\hat{X}_d(k)$ for $k, l = 0, 1, \cdots, d-1$ and*

$$\begin{aligned} \hat{X}_d(k) &= \sum_s |s\rangle\langle s+k| = \hat{X}_d(1)^k, \\ \hat{Z}_d(l) &= \sum_s e^{i2\pi sl/d}|s\rangle\langle s| = \hat{Z}_d(1)^l. \end{aligned} \tag{21}$$

The + sign denotes addition modulo d. Show that

$$\frac{1}{d^2}\sum_{m=1}^{d^2} U_m \rho U_m^\dagger = \pi, \tag{22}$$

where $\pi = \frac{I}{d}$.

4 Quantum Measurement

Quantum measurement is a process to observe the classical information within a quantum state. It can destroy the superposition property of a quantum state. The quantum measurement postulate evolves from *Born's rule* in his seminal paper in 1926, which states that "the probability density of finding a particle at a given point is proportional to the square of the magnitude of the particle's wave function at that point". Given the qubit state $|b\rangle$ in Eq. (**??**), Born's rule says that we can observe this qubit in state $|0\rangle$ with probability $|\alpha|^2$ and in state $|1\rangle$ with probability $|\beta|^2$.

Furthermore, after the measurement, the qubit state $|b\rangle$ will disappear and collapse to the observed state $|0\rangle$ or $|1\rangle$.

In general, a quantum measurement is mathematically described by a collection of $\Upsilon := \{M_i\}$, where each measurement operator $M_i \in \mathcal{L}(\mathcal{H})$ satisfies

$$\sum_i M_i = I \tag{23}$$

and each M_i is positive semi-definite operator - this means that M_i is Hermitian and all the eigenvalues are larger or equal to zero. We call this measurements positive operator-valued measure (POVM). The probability of obtaining an outcome i on a quantum state ρ is

$$p_i := \mathrm{Tr}(M_i \rho). \tag{24}$$

The state after measurement will be altered as

$$\rho_i := \frac{M_i \rho}{p_i}.$$

The normalised condition in Eq. (23) guarantees that

$$\begin{aligned}\sum_i p_i &= \sum_i \mathrm{Tr}(M_i \rho) \\ &= \mathrm{Tr}\left(\sum_i M_i \rho\right) \\ &= \mathrm{Tr}\,\rho = 1.\end{aligned} \tag{25}$$

Projective Measurement and Observables

A special instance of quantum measurements is the *projective* measurement. A projective measurement Υ is a collection of projectors $\{P_0, P_1, \cdots, P_{L-1}\}$ which sum to identity. Note that $P_i P_j = 0$ for $i \neq j$ and $P_i^2 = P_i$. When we measure a quantum state $|\phi\rangle$ with Υ, we will get the outcome j with probability

$$p_j := \mathrm{Tr}(P_j |\phi\rangle\langle\phi|)$$

and the resulting state

$$\frac{P_j|\phi\rangle}{\sqrt{p_j}}.$$

A projective measurement $\Upsilon = \{P_i\}$ with the corresponding measurement outcomes $\{\lambda_i\} \in \mathcal{R}$ can be efficiently represented by a Hermitian matrix $H = \sum_i \lambda_i P_i$. Such a matrix is called an *observable*. In physics, an observable is a physical quantity that can be measured. Examples of *observables* of a physical system include the position or momentum of a particle, among many others.

Measuring the observable H means that performing the projective measurement $\Upsilon = \{P_i\}$ on a quantum state $|\phi\rangle$. It follows that the expected value of the outcomes if we measure the state $|\phi\rangle$ with $\Upsilon = \{P_i\}$ is

$$\langle H \rangle := \sum_i \lambda_i \,\mathrm{Tr}\, P_i |\phi\rangle\langle\phi| = \langle\phi|H|\phi\rangle. \tag{26}$$

Exercise 13. *Show that every POVM can be constructed by a projective measurement on a larger Hilbert space.*

Quantum measurement can be used to distinguish a set of quantum states. We will elaborate on state distinguishability in future lectures.

Further Reading

A very good lecture note by Ronald de Wolf can be downloaded here [1].

For a better understanding of quantum channels, I would recommend [3].

References

[1] Ronald de Wolf, *Quantum computing: Lecture notes*, 2019.

[2] Christopher A. Fuchs and Carlton M. Caves, *Ensemble-dependent bounds for accessible information in quantum mechanics*, Phys. Rev. Lett. **73** (1994Dec), 3047–3050.

[3] Vinayak Jagadish and Francesco Petruccione, *An invitation to quantum channels*, arXiv preprint arXiv:1902.00909 (2019).