Methods in quantum computing

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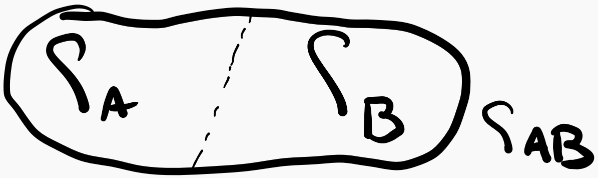
Last lecture



Purity of a quantum state is defined as $\mathrm{Tr}[\rho^2]$. Unitary operations preserve purity. We can use purity as an entanglement test. Take a pure state $\rho_{A,B} = |\psi\rangle\langle\psi|$

ρ_A, ρ_B is pure $\mathrm{Tr}[\rho_A^2] = \mathrm{Tr}[\rho_B^2] = 1$

- If there is no entanglement between A and B both $\rho_A = Tr_B \rho_{A,B}$ and $\rho_B = Tr_A \rho_{A,B}$ are pure states ($\rho_{A,B}$ was a product state).
- Otherwise $\rho_{A,B}$ is entangled. $\mathrm{Tr}[\rho_A^2] \neq 1$

This test only works if the original state was pure.

Today

1. No-cloning theorem
2. Measuring distance
3. Quantum channels
4. Noise channels
5. Measurement

No cloning theorem

Quantum states cannot be cloned

$\psi \rightarrow \psi \otimes \psi$

general, unknown state

Theorem (No-Cloning theorem)
There is no unitary operation U_{copy} on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that for all $|\psi\rangle_A \in \mathcal{H}_A$ and $|0\rangle_B \in \mathcal{H}_B$

$$U_{\text{copy}}(|\phi\rangle_A \otimes |0\rangle_B) = e^{if(\phi)}|\phi\rangle_A \otimes |\phi\rangle_B \tag{1}$$

for some number $f(\phi)$ that depends on the initial state $|\phi\rangle$.

Exercise

$|0\rangle, |1\rangle$

$\langle \phi_A | \psi_A \rangle$ = something

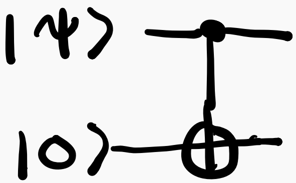
$\langle \phi | \psi \rangle = 0$ $\quad \langle \phi | \psi \rangle = 1$ $\quad$ we can copy

Prove the no-cloning theorem by contradiction.

a Assuming U_{copy} exists, take two states $|\phi_A\rangle$ and $|\psi_A\rangle$. Now apply U_{copy} on both of them and compute the resulting inner product $((\langle\phi|_A \otimes \langle 0|_B) U^{\dagger}_{\text{copy}} U_{\text{copy}}(|\psi\rangle_A \otimes |0\rangle_B))$.

$\langle \phi_A 0|$ $\quad |10\rangle \downarrow |1\rangle \otimes |0\rangle$ $\quad \cancel{\langle \phi_A | \phi_B \rangle}$

b Explain how (a) leads to a contradiction.

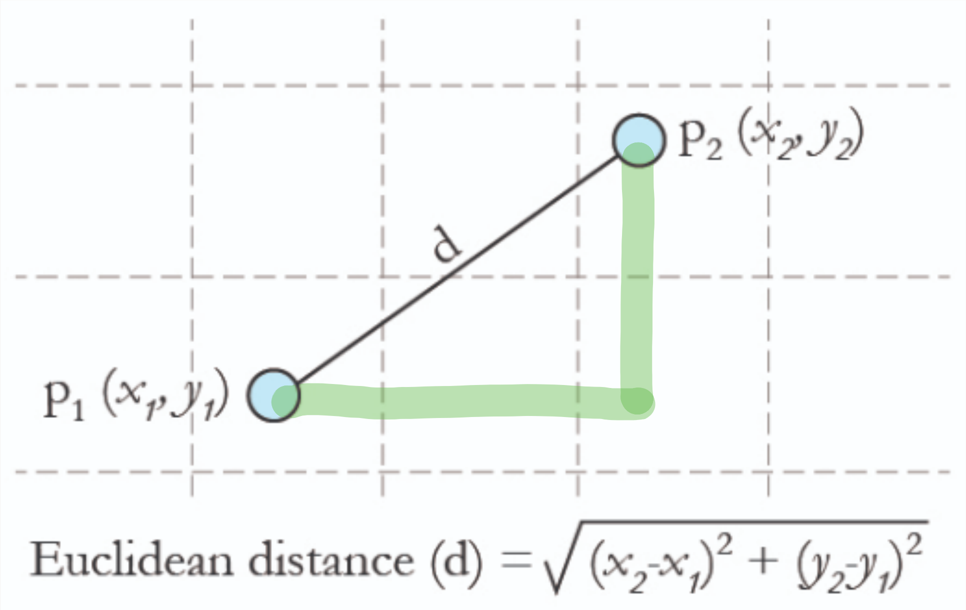


$|\psi\rangle = |0\rangle \to |0\rangle|0\rangle$

$|\psi\rangle = |1\rangle \to |11\rangle$

Distance measures

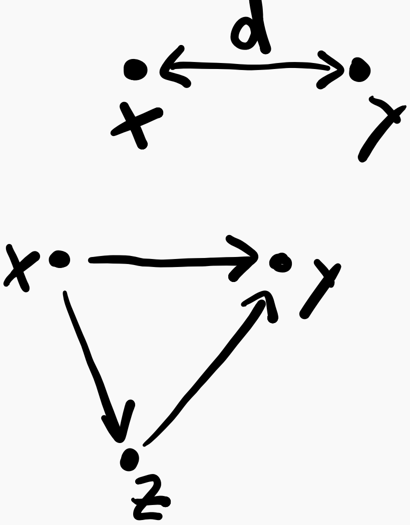
How do we measure the distance between quantum states or between quantum operators?



Metric (distance function)

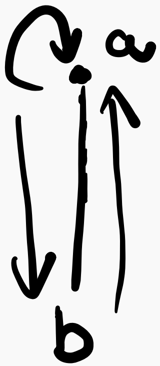
A function d can be used a metric if it satisfies the following criteria:

- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$



Example

Show that $d(a, b) \geq 0$ non-negative

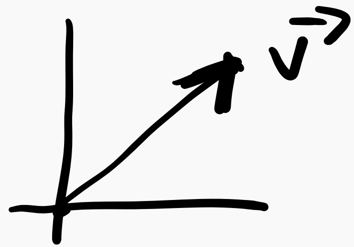


$d(a,a) = 0$ ①

3 $d(a,a) \leq d(a,b) + d(b,a)$

$$0 \leq 2 \cdot \underbrace{d(a,b)}_{\geq 0}$$

Operator Norm



Distance of the operator from 0.

every norm $\|\cdot\|$ must satisfy the following conditions.

- $\|A\| \geq 0$ with equality if and only if $A = 0$.
- $\|\alpha A\| = |\alpha|\|A\|$ for any $\alpha \in \mathbb{C}$.
- Triangle inequality: $\|A + B\| \leq \|A\| + \|B\|$.

$\mathcal{H}_A \to \mathbb{R}_+^0$

The trace distance

wiki matrix norms

Trace norm $\|A\|_1 = \mathrm{Tr}|A|$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\mathrm{Tr}(Z) = 0$

$\|Z\|_1 = 2$

U $\tilde{U}$ $\|U - \tilde{U}\|_1 \leq \varepsilon$

The *trace distance* between two operators A and B is given by

$C = A - B$

$$\|A - B\|_1 := \mathrm{Tr}\,|A - B|.$$

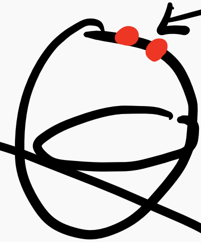
$\mathrm{Tr}|C|$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in Z basis

$X_{\text{basis}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ not a quantum state

Infidelity and fidelity

closeness



high fidelity
low infidelity

Infidelity $=$ 1- fidelity

Infidelity is a metric, fidelity is not.

For $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, their fidelity is

$$F(\rho,\sigma) := \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2 .$$

$F(\rho,\sigma) = F(\sigma,\rho)$

1

Sometime people use $\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$ as a definition

ρ or σ is pure

$F(\rho,\sigma)$

exercise: Fidelity on pure states

$\rho^2 = \rho = \sqrt{\rho}$ for pure states $\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \ldots \end{pmatrix}$

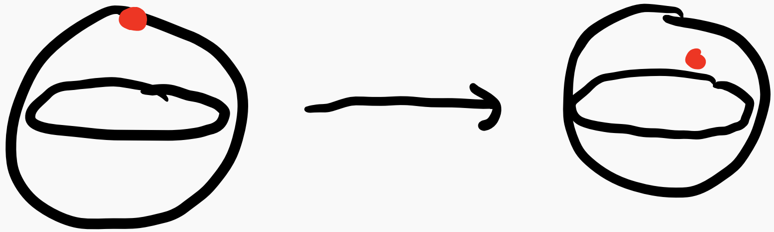
$\left(\mathrm{Tr}\sqrt{|\psi\rangle\langle\psi|\sigma|\psi\rangle\langle\psi|}\right)^2 = \langle\psi|\sigma|\psi\rangle \underbrace{\left(\mathrm{Tr}\ |\psi\rangle\langle\psi|\right)^2}_{1}$

$= \langle\psi|\sigma|\psi\rangle$

Simplify the expression for fidelity $F(\rho, \sigma) = (\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$ when $\rho = |\psi\rangle\langle\psi|$.

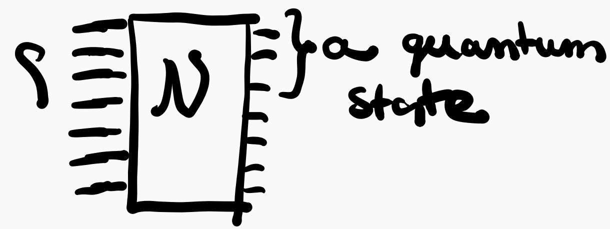
Hint: Choose a base where ρ is diagonal. How does $\sqrt{|\psi\rangle\langle\psi|}$ look like?

CPTP maps



completely positive trace preserving

Channels are the most general operation of quantum states. They must be always map quantum states onto quantum states, even if if we apply the channel only on a subset of qubits.



Any such channel can be written as

Kraus operators:
$$\Phi(\sigma) = \sum_i B_i \sigma B_i^\dagger \quad \text{where} \quad \sum_i B_i B_i^\dagger = \mathbb{1} \tag{2}$$

unitary is a special case

$$\Phi_U(\sigma) = U \sigma U^\dagger \qquad UU^\dagger = \mathbb{1}$$

set of operators $\{B_i\}$

Unitary channel $N(\rho) = U\rho U^{\dagger}$

- Depolarizing Channel:

$$\mathcal{N}(\rho) = (1-p)\rho + p\pi,$$

not expressed in terms of Kraus operators

where π is the completely mixed state $\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

- Dephasing Channel: $\sum_i B_i \rho B_i^{\dagger}$

lost info about the phase between $|0\rangle$ and $|1\rangle$

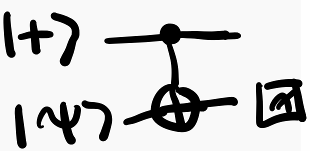
$$\mathcal{N}(\rho) = (1-p)\rho + pZ\rho Z^{\dagger}.$$

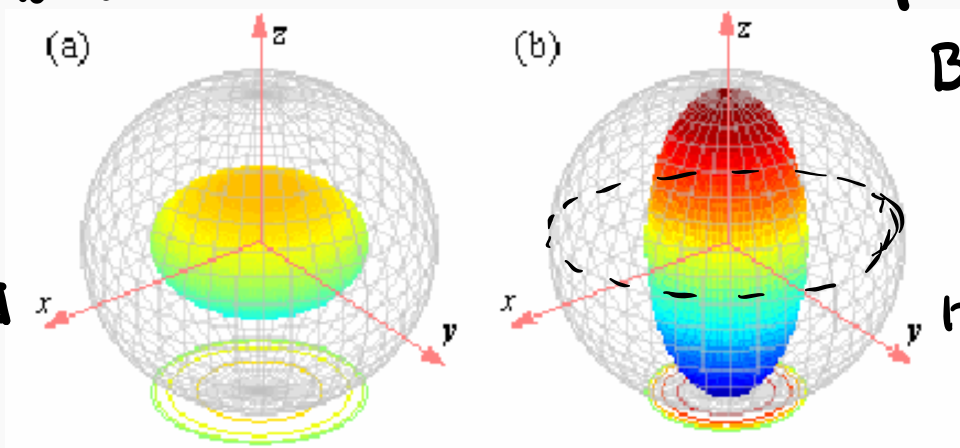
$B_0 = \mathbb{1}\cdot(1-p)$

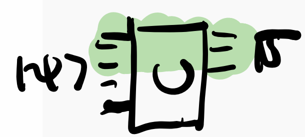
$B_1 = Z \cdot p$

$B_0 B_0^{\dagger} + B_1 B_1^{\dagger} = \mathbb{1}$

$Z^{\dagger} = Z$







Exercise

$$\rho = \left(\sqrt{\rho}\right)^2$$

$$\rho \sqrt{\rho} = \sqrt[3]{\rho} = \sqrt{\rho}\,\rho$$

A state ρ went through the depolarizing noise channel

$$\mathcal{N}(\rho) = (1 - p)\rho + p\pi,$$

where π is the completely mixed state.

What is the fidelity between ρ and $\mathcal{N}(\rho)$?

Quantum measurement

$|+\rangle$ $\quad |\ \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

Obtain classical information from a quantum state. It can destroy the superposition property of a quantum state.

Observe this qubit in state $|0\rangle$ with probability $|\alpha|^2$ and in state $|1\rangle$ with probability $|\beta|^2$. Furthermore, after the measurement, the qubit state $|b\rangle$ will disappear and collapse to the observed state $|0\rangle$ or $|1\rangle$.

$a|0\rangle + b|1\rangle$

↳ measured 0

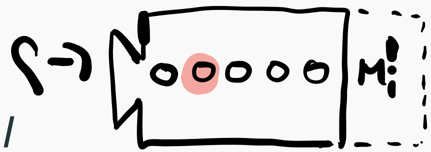
new state

$|0\rangle$

$|+\rangle$

General quantum measurement

A collection of $\Upsilon := \{M_i\}$, where each measurement operator $M_i \in \mathcal{L}(\mathcal{H})$ satisfies



probabilities of outcomes are $0 \le p_i \le 1$

$$\sum_i M_i = I \tag{3}$$

and each M_i is positive semi-definite operator. We call this measurements positive operator-valued measure (POVM). The probability of obtaining an outcome i on a quantum state ρ is

$\rho = |\psi\rangle\langle\psi|$

$\mathrm{Tr}(M_i |\psi\rangle\langle\psi|) = \mathrm{Tr}(\langle\psi|M_i|\psi\rangle)$

$$p_i := \mathrm{Tr}(M_i\rho). \tag{4}$$

$= \langle\psi|M_i|\psi\rangle$

$|0\rangle$ measure in the Z basis

$M_0 = |0\rangle\langle 0| \to i=0$

$M_1 = |1\rangle\langle 1| \to i=1$

$\sum_{i=0}^{1} M_i = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$

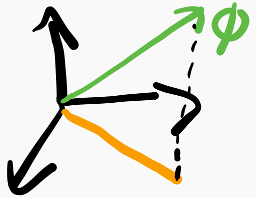
The state after measurement will be altered as

$\mathrm{Tr}(M_0\rho) = \mathrm{Tr}(|0\rangle\langle 0|0\rangle\langle 0|) = 1$

$\mathrm{Tr}(M_1\rho) = \mathrm{Tr}(|1\rangle\langle 1|0\rangle\langle 0|) = 0$

$$\rho_i := \frac{M_i\rho}{p_i}.$$

Projective measurement



measuring a pure state $|\phi\rangle$

Each M_i is a projector

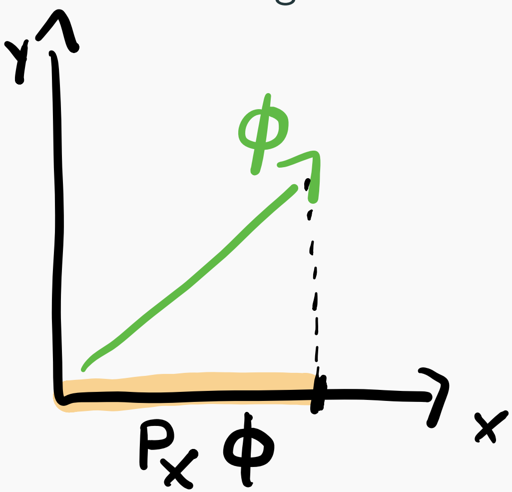
$$p_j := \mathrm{Tr}\left(P_j|\phi\rangle\langle\phi|\right)$$

$|\psi\rangle\langle\psi|$

and the resulting state

$$\frac{P_j|\phi\rangle}{\sqrt{p_j}}.$$

norm of the new state is again 1



$$\frac{|\psi\rangle\langle\psi|\phi\rangle}{\sqrt{p_j}}$$

one dim projector

Measuring observables

→ tomography

$H = \sum_i \lambda_i P_i$

Hermitian operator

Measuring the observable H means that performing the projective measurement $\Upsilon = \{P_i\}$ on a quantum state $|\phi\rangle$. It follows that the expected value of the outcomes if we measure the state $|\phi\rangle$ with $\Upsilon = \{P_i\}$ is

$$\langle H\rangle := \sum_i \lambda_i \operatorname{Tr} P_i |\phi\rangle\langle\phi| = \langle\phi|H|\phi\rangle. \tag{5}$$

$H = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

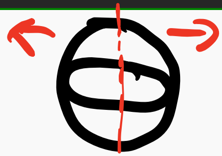
$|\phi\rangle$ is one of the eigenvector $|\psi_i\rangle$

→ single measurement

$\langle H\rangle := \mathrm{Tr}\left(\sum_i \lambda_i P_i |\phi\rangle\langle\phi|\right) = \mathrm{Tr}\left(\langle\phi|\overbrace{\sum_i \lambda_i P_i}^{H}|\phi\rangle\right)$

$= \langle\phi|H|\phi\rangle$

Expectation values



$= |+\rangle\langle+| - |-\rangle\langle-|$

Suppose we measure the operator X on the state $|0\rangle$. $X = P_+ - P_-$

What will the be outcomes of the measurement and what will be the expectation value?

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\frac{|+\rangle + |-\rangle}{\sqrt{2}} = |0\rangle \qquad \frac{|+\rangle - |-\rangle}{\sqrt{2}} = |1\rangle$$

$$\underbrace{|+\rangle\langle+|}_{P_+} \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}} \right) \qquad X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\text{exp: } \langle 0|X|0\rangle = \langle 0|0\rangle\underbrace{\langle 1|0\rangle}_{0} + \underbrace{\langle 0|1\rangle}_{=0}\langle 0|0\rangle$$

$$= 0$$